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VAN DER WAERDEN AND RAMSEY TYPE GAMES

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Let us consider the following 2-player game, called *van der Waerden game*. The players alternately pick previously unpicked integers of the interval $\{1, 2, ..., N\}$. The first player wins if he has selected all members of an n-term arithmetic progression. Let $W^*(n)$ be the least integer N so that the first player has a winning strategy.

By the *Ramsey game on k-tuples* we shall mean a 2-player game where the players alternately pick previously unpicked elements of the complete k-uniform hypergraph of N vertices K_N^k, and the first player wins if he has selected all k-tuples of an n-set. Let $R_k^*(n)$ be the least integer N so that the first player has a winning strategy.

We prove $(W^*(n))^{1/n}\to 2$, $R_2^*(n)<(2+\varepsilon)^n$ and $R_k^*(n)<2^{n^k/k!}$ for $k\geqq 3$.

1. Introduction

The finite form of van der Waerden's well-known theorem states that, for each positive integer n, there exists a smallest integer $W(n)$ with the property that if the integers from 1 to $W(n)$ are arbitrarily partitioned into two classes, then at least one class contains an arithmetic progression of n terms.

In connection with van der Waerden's theorem let us consider the following 2-player game, called *van der Waerden game*. The players alternately pick previously unpicked integers of the interval $\{1, 2, ..., N\}$ and the first player wins once he has chosen all members of an arithmetic progression of n terms. Otherwise the second player wins. Let $W^*(n)$ be the least integer N such that the first player has a winning strategy, that is, the first player can in any case cover an arithmetic progression of n terms in the interval $\{1, 2, ..., W^*(n)\}$.

The best lower bound on $W(n)$ known, due to Berlekamp [4], Erdős and Lovász (unpublished), asserts that $W(p)>p2^p$ if p is a prime and $W(n)>c2^n$ for all n. However, the best upper bound on $W(n)$ known at present is extremely poor, e.g. the following question is unsolved: is there an integer k such that $W(n)<\exp_k n$ for each n, where $\exp_k n$ denotes the k-fold iterated exp of n. Therefore it is somewhat surprising that we can determine the true order of magnitude of $W^*(n)$, namely we shall prove

$$2^{n-7n^{7/8}} < W^*(n) < n^3 2^{n-4} \tag{1.1}$$

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Secondly consider Ramsey's theorem on triples. Let K_N^3 be the family of all triples of elements of an N-element set $\mathbf{N}$. Let $R_3(n)$ be the least integer N so that if we divide K_N^3 into two classes then at least one of them contains all triples of some n-element subset of $\mathbf{N}$.

By *Ramsey game on triples* we shall mean a 2-player game where the players alternately pick previously unpicked elements of K_N^3, and the first player wins if he has selected all triples of some n-element subset of $\mathbf{N}$. Otherwise the second player wins. Let $R_3^*(n)$ be the least integer N so that the first player has a winning strategy, that is, the first player can in any case select all triples of some n-element subset.

The best estimates known for the Ramsey function $R_3(n)$ are the following [8]:

$$2^{n^2/6} < R_3(n) < 2^{2^{4n-10}}$$

and there is a large gap between the upper and lower bounds. We conjecture, but cannot prove that the true order of magnitude of $R_3^*(n)$ is about 2^{cn^2}. We shall prove, however, that

$$2^{n^2/6} < R_3^*(n) < n^4 2^{n^3/6}. \tag{1.2}$$

Here the lower bound is trivial from a theorem of Erdős and Selfridge [9].

The van der Waerden games and the Ramsey games on triples are particular cases of the following general concept, introduced by Berge [2].

First we need some terminology. A hypergraph is a collection of sets. This paper deals with finite hypergraphs only. The sets in the hypergraph are called *edges*, and the elements of these edges are *points (vertices)*. The hypergraph is r-uniform if every edge has r points. The hypergraph is said to be *almost disjoint* if any two edges have at most one point in common. Let $|H|$ denote the number of elements of H.

Let us consider a game played by two players on a hypergraph $\mathcal{F}$ such that the players alternately pick previously unpicked points of $\mathcal{F}$. The game is called a *positional game of first type* if the first player wins whenever he picks all points of some $A\in\mathcal{F}$, otherwise the second player is the winner.

Berge has also defined *positional games of second type*. In such a game that player wins who picks all points of some $A\in\mathcal{F}$ first; if all points are occupied then the game is a tie. The traditional Tick-Tack-Toe game gives an example where the first player has a winning strategy in the positional game of first type, but the second player has a drawing strategy in the positional game of second type. (Tick-Tack-Toe is played on a 3×3 array of points in the plane, see [11].)

We shall investigate positional games of first type only. Let $G(\mathcal{F})$ denote the positional game of first type on the hypergraph $\mathcal{F}$.

The following general results are known [9], [1]:

Theorem A. *If $\mathcal{F}$ is an r-uniform hypergraph and $|\mathcal{F}|<2^{r-1}$, then the second player has a winning strategy in $G(\mathcal{F})$.*

Theorem B. *If $\mathcal{F}$ is an almost disjoint r-uniform hypergraph and $|\mathcal{F}|<4^{r-c\sqrt{r}}$ (c is an absolute constant), then the second player has a winning strategy in $G(\mathcal{F})$.*

In fact, it has been proved in [9] and also in [1] that under these assumptions the second player has a drawing strategy in the positional game of second type, but the proofs give these stronger theorems without any modification.

The upper bounds in (1.1) and (1.2) are immediate consequence of the following theorem.

Let $S(\mathcal{F})$ denote the set of points of $\mathcal{F}$. Let $d_{x_1,x_2}(\mathcal{F})$ denote the number of edges of $\mathcal{F}$ containing x_1 and $x_2 \in S(F)$, and let $d_2(\mathcal{F})$ denote the maximum of $d_{x_1,x_2}(\mathcal{F})$ for all pairs x_1, x_2 $(x_1 \neq x_2)$.

Theorem 1. *Let $\mathcal{F}$ be an r-uniform hypergraph. If*

$$|\mathcal{F}| > 2^{r-3} d_2(\mathcal{F})|S(\mathcal{F})|,$$

then the first player has a winning strategy in $G(\mathcal{F})$.

It seems hopeless to prove a similar theorem on positional games of second type. In this case we know only of the following sufficient condition (see [2]):

(1.3) *if the chromatic number of $\mathcal{F}$ is at least 3 then the first player has a winning strategy in the positional game of second type on $\mathcal{F}$.*

Remark that (1.3) easily follows from the well-known Zermelo—von Neumann theorem. Unfortunately, (1.3) is very inconvenient for the most part of applications.

Theorem 2.

$$\sqrt[n]{W^*(n)} \to 2.$$

We mention that Theorem A immediately implies $\liminf \sqrt[n]{W^*(n)} \geqq \sqrt{2}$. The lower bound $\liminf \sqrt[n]{W^*(n)} \geqq 2$ could be obtained directly from Theorem B if the family of arithmetic progressions of length n were almost disjoint. Unfortunately this family is not almost disjoint, however, in some sense it is "nearly" almost disjoint. This observation will be basic in the proof. Now we define the *Ramsey game on k-tuples* for $k \geqq 2$ as follows. Let K_N^k be the complete k-uniform hypergraph on an N-element set **N**, and the players alternately pick previously unpicked elements of K_N^k. The first player wins if he can choose every k-tuples of some n-element subset of **N**, or else the second player wins. Let $R_k^*(n)$ be the least N so that the first player has a winning strategy in this game.

Theorem 3.

$$2^{\frac{n}{2}} < R_2^*(n) < (2+\varepsilon)^n, \tag{1.4}$$

$$2^{\frac{n^{k-1}}{k!}} < R_k^*(n) < n^{k+1} 2^{\binom{n}{k}} \quad \textit{for} \quad k \geqq 3. \tag{1.5}$$

Here the lower bounds are trivial consequences of Theorem A (see [9]). The proof of the upper bound in (1.4) is a combination of the standard "ramification method" and the "weight function" method. The upper bound in (1.5) follows from Theorem 1.

Finally we mention that several particular classes of the positional games of first type have been investigated in the literature, e.g. "Bridge it" (Shannon switching game) in [12], [7], [5], [6], "Match it" in [3], Hex in [10], etc.

2. Proof of Theorem 1

We shall use the "weight function" method. Suppose that the first and the second player have picked the points $x_1, \ldots, x_i$ and $y_1, \ldots, y_i$, respectively. Now we define the hypergraph. $\mathcal{F}_i$ as follows: throw away the edges of $\mathcal{F}$ blocked by the points $y_1, y_2, \ldots, y_i$ and from the remaining edges throw away the points $x_1, x_2, \ldots, x_i$, i.e. let

$$\mathcal{F}_i = \left\{A \setminus \{x_1, \ldots, x_i\} \colon A \in \mathcal{F} \text{ and } A \cap \{y_1, \ldots, y_i\} = \emptyset\right\}.$$

We give each edge B of $\mathcal{F}_i$ the value $2^{-|B|}$. Give each point x of $\mathcal{F}$ the value $\varphi_i(x)$ which is the sum of the values of each edge of $\mathcal{F}_i$ it belongs to.

Here is the winning strategy of the first player: in the $i+1$-st move pick a point of largest value.

In order to prove that this is indeed a winning strategy, we show that the sum φ_i of the values of all edges of $\mathcal{F}_i$ is positive for every i. To show this assertion we require the following inequality

$$\varphi_{i+1} \geqq \varphi_i - \frac{d_2(\mathcal{F})}{4}. \tag{2.1}$$

For verifying (2.1) observe that on the first player's $i+1$-st move he doubles the value of each edge containing the point picked, i.e. he adds $\varphi_i(x_{i+1})$ to φ_i. After the $i+1$-st move of the second player the sum φ_{i+1} of the values of all sets of $\mathcal{F}_{i+1}$ is equal to

$$\{\varphi_i + \varphi_i(x_{i+1})\} - \varphi_i(y_{i+1}) - \psi_i$$

where

$$\psi_i = \sum_{B \in \mathcal{F}_i \colon \{x_{i+1}, y_{i+1}\} \subset B} 2^{-|B|}.$$

It is easy to see that $\psi_i \leqq \dfrac{d_2(\mathcal{F})}{4}$, therefore

$$\varphi_{i+1} \geqq \{\varphi_i + \varphi_i(x_{i+1})\} - \varphi_i(y_{i+1}) - \psi_i \geqq$$

$$\geqq \varphi_i - \psi_i \geqq \varphi_i - \frac{d_2(\mathcal{F})}{4},$$

since $\varphi_i(x_{i+1})$ was a maximum. From (2.1) follows

$$\varphi_i \geqq \varphi_{i-1} - \frac{d_2(\mathcal{F})}{4} \geqq \varphi_0 - i\,\frac{d_2(\mathcal{F})}{4}.$$

On the other hand,

$$\varphi_0 = \frac{|\mathcal{F}|}{2^r} \quad \text{and} \quad i \leqq \frac{|S(\mathcal{F})|}{2},$$

so we have

$$\varphi_i \geqq |\mathcal{F}|\,2^{-r} - \frac{|S(\mathcal{F})|\, d_2(\mathcal{F})}{8} > 0,$$

which was to be proved. ∎

3. Universal winning strategies

Let $\mathcal{F}$ be a hypergraph and m be a natural number, we define a 2-player combinatorial game $G(\mathcal{F}, m)$ as follows. The players alternately pick previously unpicked points of $S(\mathcal{F})$ (we remind that $S(\mathcal{F})$ denotes the set of points of $\mathcal{F}$) and the first player wins if he picked m points of some $A \in \mathcal{F}$ and up to that time the second player has picked no point of A. Otherwise the second player wins.

We need also the following generalization of the game $G(\mathcal{F}, m)$. Let $S_1 \supseteqq S_2 \supseteqq \dots$ be a decreasing sequence of the subsets of $S(\mathcal{F})$ and the new rules are that the second player in his i-th move and the first player in his $i+1$-st move are obliged to pick an unpicked point of S_i, i.e. $\{y_i, x_{i+1}\} \subset S_i$, $i=1, 2, \dots$. The first wins, similarly as before, if he picked m points of some $A \in \mathcal{F}$ and up to that time the second has picked no point of A. Otherwise the second wins. Denote this game by $G(\mathcal{F}, \{S_i\}_i, m)$.

We call a function $f=f(\mathcal{F})$: $2^S \times 2^S \times 2^S \to S$ (where $S=S(\mathcal{F})$) a *universal strategy for the second player on* $(\mathcal{F}, S(\mathcal{F}))$ if for each decreasing sequence $\{S_i\}_i$ $(S(\mathcal{F}) \supseteqq S_1 \supseteqq S_2 \supseteqq \dots)$:

$$y_i = f(S_i, \{x_1, \dots, x_i\}, \{y_1, \dots, y_{i-1}\}) \in S_i \setminus \{x_1, \dots, x_i, y_1, \dots, y_{i-1}\}$$

if

$$S_i \setminus \{x_1, \dots, x_i, y_1, \dots, y_{i-1}\} \neq \emptyset.$$

Here $x_1, x_2, \dots$ denote the first player's points, and 2^S denotes the family of subsets of S.

Lemma 1. *The second player has a universal strategy $f_1 = f_1(\mathcal{F})$ on $(\mathcal{F}, S(\mathcal{F}))$, such that if $|\mathcal{F}| < 2^{m-1}$ and $\min_{A \in \mathcal{F}} |A| \geqq m$, then the choices*

$$y_i = f_1(S_i, \{x_1, \dots, x_i\}, \{y_1, \dots, y_{i-1}\})$$

guarantee for him the win in all possible games $G(\mathcal{F}, \{S_i\}_i, m\}$.

Observe that Lemma 1 is a generalization of Theorem A.

Proof. Our argument will be a slight modification of the Erdős-Selfridge proof (see [9]). Suppose that the first and the second player previously picked $x_1, \dots, x_i$ and $y_1, \dots, y_{i-1}$, respectively, and we wish to choose a good point $y_i \in S_i$ for the second. We define $\mathcal{F}_i$ as follows:

$$\mathcal{F}_i = \{A \setminus \{x_1, \dots, x_i\} \colon A \in \mathcal{F} \text{ and } A \cap \{y_1, \dots, y_{i-1}\} = \emptyset\}.$$

For every $A \in \mathcal{F}$ let $\varrho_i(A) = \infty$ if $A \cap \{y_1, \dots, y_{i-1}\} \neq \emptyset$ and $\varrho_i(A) = m - |A \cap \{x_1, \dots, x_i\}|$ if $A \cap \{y_1, \dots, y_{i-1}\} = \emptyset$. Give each point $x \in S(\mathcal{F})$ the value $\varphi_i(x) = \sum_{x \in A \in \mathcal{F}} 2^{-\varrho_i(A)}$ and let $\varphi_i = \sum_{A \in \mathcal{F}} 2^{-\varrho_i(A)}$.

Here is the desired universal strategy: in the i-th move pick from $S_i \setminus \{x_1, \dots, x_i, y_1, \dots, y_{i-1}\}$ a point of largest value. We claim

$$\varphi_{i+1} \leqq \varphi_i. \tag{3.1}$$

Indeed, observe that $\varphi_{i+1} \leqq \varphi_i - \varphi_i(y_i) + \varphi_i(x_{i+1}) \leqq \varphi_i$, since $\{y_i, x_{i+1}\} \subset S_i$ and $(\varphi_i y_i)$ was a maximum in S_i. (3.1) yields $\varphi_i \leqq \varphi_1$ for all i. By definition $\varphi_1 \leqq |\mathcal{F}| 2^{-m+1}$ therefore, if $|\mathcal{F}| < 2^{m-1}$ then we get $\varphi_i < 1$ for all i. Now assume in contrary that $|\mathcal{F}| < 2^{m-1}$ and the first player can win in the game $G(\mathcal{F}, \{S_i\}_i, m)$. Then there exists $A^* \in \mathcal{F}$ such that for an appropriate i $A^* \subset \{x_1, \ldots, x_i\}$, therefore by the hypothesis of the lemma $\varphi_i \geqq 2^{|\{x_1, \ldots, x_i\} \cap A^*| - m} = 2^{|A^*| - m} \geqq 1$, a contradiction. So Lemma 1 is proved. ∎

Corollary. *The second player has a strategy $f_2 = f_2(\mathcal{F})$: $2^S \times 2^S \to S$ (where $S = S(\mathcal{F})$) such that, if $|\mathcal{F}| < 2^{m-1}$ and $\min_{A \in \mathcal{F}} |A| \geqq m$, then the choices $y_i = f_2(\{x_1, \ldots, x_i\}, \{y_1, \ldots, y_{i-1}\})$ guarantee for him the win in $G(\mathcal{F}, m)$.*

Proof. Let $S(\mathcal{F}) = S_1 = S_2 = \ldots$ and apply Lemma 1. ∎

We shall find useful to consider a further generalization of the concept of positional games of first type.

We say that $\mathcal{H}$ is a *truncated subsystem* of $\mathcal{F}$ if each edge of $\mathcal{H}$ is contained in one of the edges of $\mathcal{F}$.

Consider a hypergraph $\mathcal{F}$ and an increasing sequence $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \ldots$ of truncated subsystems of $\mathcal{F}$. We define a 2-player combinatorial game $G(\mathcal{F}, \{\mathcal{H}_i\}_i, m)$ as follows: The players in their i-th move pick previously unpicked points of $S(\mathcal{H}_i)$ for $i = 1, 2, \ldots$, and the first player wins if he has picked m points of some $A \in \mathcal{F}$ and up to that time the second player has picked no point of A. Otherwise the second player wins.

Let $\mathcal{R}(\mathcal{F})$ denote the family of all truncated subsystems of $\mathcal{F}$.

We say that the function $f = f(\mathcal{F})$: $\mathcal{R} \times 2^S \times 2^S \to S$ (where $\mathcal{R} = \mathcal{R}(\mathcal{F})$ and $S = S(\mathcal{F})$) is a *universal strategy of the second player on* $(\mathcal{F}, \mathcal{R}(\mathcal{F}))$ if for each increasing sequence $\{\mathcal{H}_i\}_i$ (where $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \ldots$ are elements of $\mathcal{R}(\mathcal{H})$):

$$y_i = f(\mathcal{H}_i, \{x_1, \ldots, x_i\}, \{y_1, \ldots, y_{i-1}\}) \in S(\mathcal{H}_i) \setminus \{x_1, \ldots, x_i, y_1, \ldots, y_{i-1}\}$$

if

$$S(\mathcal{H}_i) \setminus \{x_1, \ldots, x_i, y_1, \ldots, y_{i-1}\} \neq \emptyset.$$

Here $x_1, x_2, \ldots$ denote the first player's points.

We say that a truncated subsystem $\mathcal{H}$ of $\mathcal{F}$ is of type (ν, μ) if every $A \in \mathcal{F}$ meets at most ν edges of $\mathcal{H}$ and for all $B \in \mathcal{H}$ there are at most μ edges of $\mathcal{F}$ having at least two common points with B.

Lemma 2. *The second player has such a universal strategy $f_3 = f_3(\mathcal{F})$ on $(\mathcal{F}, \mathcal{R}(\mathcal{F}))$ that, if $\nu \leqq m^{6/7}$, $\mu < 2^{[m]^{1/7} - 1}$ and $m \geqq 128$, then the second player wins by f_3 in the games $G(\mathcal{F}, \{\mathcal{H}_i\}_i, m)$, where $\{\mathcal{H}_i\}_i$ is extended over all possible increasing sequences of $\mathcal{R}(\mathcal{F})$ of type (ν, μ).*

Proof. Suppose that the first player picked $x_1 \in S(\mathcal{H}_1)$. Let $\mathcal{H}_1 = \{B_1, B_2, \ldots, B_l\}$. We define a partition $\mathcal{B}_1$ of $S(\mathcal{H}_1)$ as follows: $\mathcal{B}_1 = \{B_1^*, B_2^*, \ldots, B_l^*\}$ where $B_j^* = B_j \setminus \bigcup_{i=1}^{j-1} B_i$. For each $B_j^* \in \mathcal{B}_1$ let $\mathcal{F}_j = \{A \cap B_j^*\colon A \in \mathcal{F}$ and $|A \cap B_j^*| \geqq [m^{1/7}]\}$, where $[\beta]$ denotes the integer part of the real number β. If $x_1 \in B_j^*$ then the second player chooses $y_1 = f_2(\mathcal{F}_j, \{x_1\}, \emptyset)$, where f_2 is defined in the Corollary of Lemma 1.

Now assume in general that the first and the second player previously picked the points $x_1, \dots, x_i$ and $y_1, \dots, y_{i-1}$, respectively, and the partition $\mathcal{B}_{i-1}$ and the hypergraphs $\mathcal{F}_j$ for $1 \leq j \leq |\mathcal{H}_{i-1}|$ are defined. Let $\mathcal{H}_i \setminus \mathcal{H}_{i-1} = \{B_j: |\mathcal{H}_{i-1}| < j \leq |\mathcal{H}_i|\}$. We define the partition $\mathcal{B}_i$ of $S(\mathcal{H}_i) \setminus S(\mathcal{H}_{i-1})$ and the hypergraphs $\mathcal{F}_j$ for $|\mathcal{H}_{i-1}| < j \leq |\mathcal{H}_i|$ as follows:

$$\mathcal{B}_i = \{B_j^*: |\mathcal{H}_{i-1}| < j \leq |\mathcal{H}_i|\} \quad \textit{where} \quad B_j^* = B_j \setminus \bigcup_{i=1}^{j-1} B_i,$$

$$\mathcal{F}_j = \{A \cap B_j^*: A \in \mathcal{F} \textit{ and } |A \cap B_j^*| \geq [m^{1/7}]\}.$$

Here is the desired universal strategy: if $x_i \in B_j^*$, then the second player chooses

$$y_i = f_2(\mathcal{F}_j, \{x_1, \dots, x_i\} \cap B_j^*, \{y_1, \dots, y_{i-1}\} \cap B_j^*).$$

It is easy to see that the definition of y_i is sound. We shall prove that, if the second player uses the universal strategy defined just now, moreover, $\nu \leq m^{6/7}$, $\mu < 2^{[m^{1/7}]-1}$ and $m \geq 128$, then the second player wins the game $G(\mathcal{F}, \{\mathcal{H}_i\}_i, m)$, where $\{\mathcal{H}_i\}_i$ is an arbitrary (ν, μ)-type increasing sequence of $\mathcal{R}(\mathcal{F})$. This means that for all $A \in \mathcal{F}$ either the first player cannot cover m points of A or the set of the second player's points meets A.

Since for every $B \in \bigcup_i \mathcal{H}_i$ there are less than $2^{[m^{1/7}]-1}$ edges of $\mathcal{F}$ having at least $[m^{1/7}] \geq 2$ common points with B ($[m^{1/7}] \geq 2$ if $m \geq 128$ and $\mu < 2^{[m^{1/7}]-1}$), therefore $|\mathcal{F}_j| < 2^{[m^{1/7}]-1}$ for all j. Applying Corollary we obtain that for every $C \in \mathcal{F}_j$ either the first player cannot cover $[m^{1/7}]$ points of C or the second player can pick at least one point of C. Lemma 2 follows from this, since every edge of $\mathcal{F}$ meets at most ν edges of $\bigcup_i \mathcal{H}_i$ and $\nu \leq m^{6/7}$. ∎

Lemma 3. *Given a hypergraph $\mathcal{F}$ and a real number α, $1/2 < \alpha \leq 1$, consider the following 2-player combinatorial game $G(\mathcal{F}, \alpha)$. The players alternately pick previously unpicked points of $S(\mathcal{F})$ and the second player wins if he can cover at least $\alpha|A|$ points of some $A \in \mathcal{F}$. Otherwise the first player wins. If*

$$\sum_{A \in \mathcal{F}} c_\alpha^{-|A|} < 1, \quad \textit{where} \quad c_\alpha = 2\alpha^\alpha(1-\alpha)^{1-\alpha},$$

then the first player has a winning strategy in the game $G(\mathcal{F}, \alpha)$.

Proof. The proof will be quite similar to the proof of Lemma 1, but we shall use different valuation method. Suppose that the first and the second players previously picked $x_1, \dots, x_i$ and $y_1, \dots, y_i$, respectively, and we wish to choose a good point for the first player. Now we give each edge $A \in \mathcal{F}$ a value $\varphi_i(A)$ which is equal to

$$(1+\mu)^{|A \cap \{y_1, \dots, y_i\}| - \alpha|A|} (1-\mu)^{|A \cap \{x_1, \dots, x_i\}| - (1-\alpha)|A|}$$

where $\mu = 2\alpha - 1$. We give each point $x \in S(\mathcal{F})$ a value $\varphi_i(x)$, where

$$\varphi_i(x) = \sum_{A:\, x \in A \in \mathcal{F}} \varphi_i(A).$$

Finally, let

$$\varphi_i = \sum_{A \in \mathcal{F}} \varphi_i(A).$$

Here is the desired winning strategy of the first player: in the $(i+1)$-st move pick a point of largest value. We claim $\varphi_{i+1} \leqq \varphi_i$. Indeed, observe that

$$\varphi_{i+1} \leqq \varphi_i - \mu\varphi_i(x_{i+1}) + \mu\varphi_i(y_{i+1}) \leqq \varphi_i,$$

since $\varphi_i(x_{i+1})$ was a maximum. This implies $\varphi_i \leqq \varphi_0$ for all i. By the hypothesis of the lemma

$$\varphi_0 = \sum_{A\in\mathcal{F}} (1+\mu)^{-\alpha|A|}(1-\mu)^{-(1-\alpha)|A|} = \sum_{A\in\mathcal{F}} c_\alpha^{-|A|} < 1,$$

therefore $\varphi_i < 1$ for all i.

Now, assume that the second player can cover at least $\alpha|A|$ points of some $A\in\mathcal{F}$. From this it follows that $1 \leqq \varphi_i(A) \leqq \varphi_i$ for a large enough i. Hence, our indirect assumption leads to contradiction. ▮

4. A Lemma on arithmetic progressions

Lemma 4. *Let us be given k ($k \geqq 13^5$) arithmetic progressions $P_1, \ldots, P_k$ of length n having the property that for every two progressions P_i, P_j either $P_i \cap P_j = \emptyset$ or P_i and P_j have distinct differences. Then one can select a subsequence $P_{i_1}, P_{i_2}, \ldots, P_{i_t}$ where $t=[k^{1/5}]$, such that $\left|\bigcup_{j=1}^{t} P_{i_j}\right| \geqq tn-n$.*

Proof. Let d_i denote the difference of P_i, $i=1, \ldots, k$. We may assume that $d_1 \leqq \leqq d_2 \leqq \ldots \leqq d_k$.

We start with the following simple observation:

$$\text{(4.1)} \qquad \text{if} \quad d_i \leqq d_j \leqq \left(1+\frac{1}{l}\right) d_i, \quad \text{then} \quad |P_i \cap P_j| < \frac{2n}{l} \quad (i \neq j).$$

Indeed, if $d_i = d_j$, then by the hypothesis of the lemma $P_i \cap P_j = \emptyset$, so we may assume $d_i < d_j$ and $P_i \cap P_j \neq \emptyset$. Let D denote the difference of the arithmetic progression $P_i \cap P_j$. Since d_i and d_j are divisors of D, so D can be written in the forms: $D = pd_i = qd_j$. From this follows

$$1 < \frac{d_j}{d_i} = \frac{p}{q} \leqq 1 + \frac{1}{l},$$

therefore $q \geqq l$. Now we are ready, since $|P_i \cap P_j| < \dfrac{2n}{q}$.

For notational convenience let $t=[k^{1/5}]$.

We distinguish two cases according to as Q is less than t or greater than or equal to t, where

$$Q = \max_i \left| \left\{ j\colon d_i \leqq d_j \leqq \left(1+\frac{1}{t^2}\right) d_i \right\} \right|.$$

Case 1. $Q < t$.

In this case we get for all i and j ($1 \leqq i < j \leqq k/t$):

$$d_{jt} > \left(1+\frac{1}{t^2}\right)^{j-i} d_{it}.$$

From this follows $|P_{it^4}\cap P_{jt^4}|\leqq n/k$, where $1\leqq i<j\leqq k/t^4$, since

$$\frac{d_{jt^4}}{d_{it^4}} > \left(1+\frac{1}{t^2}\right)^{(j-i)t^3} \geqq \left(1+\frac{1}{t^2}\right)^{t^3} > t^5 = k \quad \text{if} \quad t \geqq 13.$$

Therefore we have

$$\left|\bigcup_{i=1}^{t} P_{it^4}\right| \geqq tn-\binom{t}{2}\frac{n}{k} > tn-n.$$

Case 2. $Q\geqq t$.

This means that there is a suffix i such that

$$d_i \leqq d_{i+1} \leqq \ldots \leqq d_{i+t-1} \leqq \left(1+\frac{1}{t^2}\right) d_i.$$

By (4.1) we get

$$|P_{j_1}\cap P_{j_2}| < \frac{2n}{t^2}, \quad i \leqq j_1 < j_2 \leqq i+t-1,$$

therefore

$$\left|\bigcup_{j=i}^{i+t-1} P_j\right| \geqq tn-\binom{t}{2}\frac{2n}{t^2} \geqq tn-n,$$

which was to be proved. ∎

5. Proof of Theorem 2

We start with the upper bound on $W^*(n)$. We shall explicitly prove that $W^*(n)\leqq n^3 2^{n-4}$.

Let $\mathscr{P}(N, n)$ be the set of the arithmetic progressions of length n in the interval $\{1, 2, \ldots, N\}$. A simple computation shows that

$$|\mathscr{P}(N, n)| > \frac{N^2}{4(n-1)}.$$

It is easy to see that at most $\binom{n}{2}$ arithmetic progressions of length n can contain two given integers, therefore $d_2(\mathscr{P}(N, n))\leqq\binom{n}{2}$. Let $N=n^3 2^{n-4}$, then

$$\frac{N^2}{4(n-1)} \geqq 2^{n-3}\binom{n}{2} N,$$

so we may apply Theorem 1, which implies $W^*(n)\leqq n^3 2^{n-4}$.

Now we turn to the lower bound of $W^*(n)$. Let $d(P)$ denote the difference of the arithmetic progression P. Let $\tilde{\mathscr{P}}=\tilde{\mathscr{P}}(N, n)$ denote the set of arithmetic progressions having the form $\{a, a+d, \ldots, a+(n-1)d\}$, where $a\geqq 1$, $a+(n-1)d\leqq N$ and

(5.1) $$2[n^{7/8}] \text{ is a divisor of } \left[\frac{a}{d}\right].$$

We shall prove that the second player has a winning strategy in the game $G(\tilde{\mathcal{P}})=G(\tilde{\mathcal{P}}(N,n))$, where n is large enough and $N\leqq 2^{n-5n^{7/8}}$. From this $\liminf \sqrt[n]{W^*(n)}\geqq 2$ directly follows, since every arithmetic progression of length $n+2[n^{7/8}]$ in the interval $\{1,2,\dots,N\}$ contains an element of $\tilde{\mathcal{P}}$.

For a later purpose observe that:

(5.2) *if $P_1, P_2, \dots, P_r$ are different elements of $\tilde{\mathcal{P}}$ such that $d(P_1)=d(P_2)=\dots=d(P_r)$ and $P_1\cap P_i\neq\emptyset$ for all i, then $r<[n^{1/8}]$.*

Indeed, (5.2) is an easy consequence of (5.1).

Say that the subset $\{P_1,\dots,P_l\}\subset\tilde{\mathcal{P}}$ *is connected by* $P\in\tilde{\mathcal{P}}\setminus\{P_1,P_2,\dots,P_l\}$ if the sets $P\cap P_i$, $i=1,2,\dots,l$ are non-empty. For notational convenience let $t=[n^{1/8}]$. Let

$$\tilde{\mathcal{Q}}=\tilde{\mathcal{Q}}(N,n)=\left\{\bigcup_{i=1}^{t}P_i\colon P_i\in\tilde{\mathcal{P}},\ \left|\bigcup_{i=1}^{t}P_i\right|\geqq tn \text{ and } \{P_1,P_2,\dots,P_t\} \text{ is connected}\right\}.$$

Assume that n is large enough and $N=[2^{n-5n^{7/8}}]$. Now we define the desired winning strategy of the second player in $G(\tilde{\mathcal{P}})$ by recursion.

Let $r(0)=b(0)=0$, $S_0=S(\tilde{\mathcal{Q}})$ and $\tilde{\mathcal{P}}_0=\mathcal{G}_0=\emptyset$. Let $r=r(i-1)$ and $b=b(i-1)$, and suppose that we have defined the sequences $\{x_1,\dots,x_r\}$ and $\{y_1,\dots,y_r\}$ of red points; $\{v_1,\dots,v_b\}$ and $\{z_1,\dots,z_b\}$ of blue points, the hypergraphs $\tilde{\mathcal{P}}_0,\dots,\tilde{\mathcal{P}}_r$ and $\mathcal{G}_0,\dots,\mathcal{G}_r$, and the sets $S_0,\dots,S_r$. We distinguish two cases.

Case 1. The first player's i-th point is in S_r. Then let $r(i)=r+1, b(i)=b$ and $x_{r+1}\in S_r$ be the i-th move of the first. Put

$$\tilde{\mathcal{P}}_{r+1}=\{P\setminus\{x_1,\dots,x_{r+1}\}\colon P\in\tilde{\mathcal{P}},\quad P\cap\{y_1,\dots,y_r\}=\emptyset\},$$

$$\mathcal{G}_{r+1}=\{A\in\tilde{\mathcal{P}}_{r+1}\colon |A|=[n^{7/8}]\},$$

$$S_{r+1}=S(\tilde{\mathcal{Q}})\setminus S(\mathcal{G}_{r+1}),$$

$$y_{r+1}=f_1(S_{r+1},\{x_1,\dots,x_{r+1}\},\{y_1,\dots,y_r\})\in S_{r+1}$$

and let the second player's response be y_{r+1}.

Case 2. The first player's i-th point is not in S_r. Then let $r(i)=r$, $b(i)=b+1$, $v_{b+1}\in S(\mathcal{G}_r)$ be the i-th move of the first player. Put

$$z_{b+1}=f_3(\mathcal{G}_r,\{v_1,\dots,v_{b+1}\},\{z_1,\dots,z_b\})$$

and let the second player's move be z_{b+1}. The construction of the strategy is completed.

By definition $\mathcal{G}_r$ is a truncated subsystem of $\tilde{\mathcal{P}}$, therefore in order to prove that the construction above is sound it remains to check that $\mathcal{G}_r\subseteqq\mathcal{G}_{r+1}$ and $S_r\supseteqq S_{r+1}$. But this is clear, since

$$\mathcal{G}_{r+1}=\mathcal{G}_r\cup\{P'\in\tilde{\mathcal{P}}_r\colon |P'|=[n^{7/8}]+1,\ y_r\notin P' \text{ and } x_{r+1}\in P'\},$$

consequently

$$S_r = S(\tilde{\mathcal{Q}})\setminus S(\mathcal{G}_r) \supseteqq S_{r+1} = S(\tilde{\mathcal{Q}})\setminus S(\mathcal{G}_{r+1}),$$

which was to be checked.

For a later purpose we remark that $\{x_{i+1}, y_i\}\subset S_i$.

Let $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ be the red points; $v_1, \ldots, v_q$ and $z_1, \ldots, z_q$ be the blue points.

$\mathcal{G}_r\subseteqq\mathcal{G}_{r+1}$ implies $|P\cap\{x_1, \ldots, x_p\}|\leqq n-[n^{7/8}]$ for all $P\in\tilde{\mathcal{P}}$, therefore in order to check that the second player wins the game $G(\tilde{\mathcal{P}})$ it suffices to prove that $\mathcal{G}_r$ is a truncated subsystem of type (ν, μ) of $\tilde{\mathcal{P}}$, where $\nu\leqq[n^{3/4}]$ and $\mu<2^{[n^{1/8}]-1}$. Indeed, restricting to the blue points, the players are playing the game $G(\tilde{\mathcal{P}}, \{\mathcal{H}_i\}_i, [n^{7/8}])$, where $\mathcal{H}_i=\mathcal{G}_{r(i)}$, therefore Lemma 2 implies that under the conditions $\nu\leqq[m^{6/7}]$ and $\mu<2^{[m^{1/7}]-1}$, where $m=[n^{7/8}]$, either $|P\cap\{v_1, \ldots, v_q\}|<[n^{7/8}]$ or $P\cap\{z_1, \ldots, z_q\}\neq\emptyset$ for all $P\in\tilde{\mathcal{P}}$. In the first case

$$|P\cap(\{x_1, \ldots, x_p\}\cup\{v_1, \ldots, v_q\})| < (n-[n^{7/8}])+[n^{7/8}] = n,$$

that is, the first player is unable to cover P. In the other case the second player can pick a point of P, therefore the first player is again unable to cover P.

Let $\mathcal{P}^*$ be an arbitrary family of arithmetic progressions of length n, then $d_2(\mathcal{P}^*)\leqq\binom{n}{2}$, i.e. there are at most $\binom{n}{2}$ elements of $\mathcal{P}^*$ containing two given integers. From this follows that for every $P\in\mathcal{P}^*$ there are at most $\binom{n}{2}^2$ elements of $\mathcal{P}^*$ having at least two common points with P, therefore $\mu\leqq\binom{n}{2}^2$. Clearly $\binom{n}{2}^2<2^{[n^{1/8}]-1}$ if n is large enough, thus $\mu<2^{[n^{1/8}]-1}$ is checked.

Now suppose that $\nu\leqq[n^{3/4}]$ is not true. Then we can choose $P_0\in\tilde{\mathcal{P}}$ and $A_1, \ldots, A_\nu\in\mathcal{G}_p$ for $\nu\geqq n^{3/4}$ such that $P_0\cap A_i\neq\emptyset$, $i=1, \ldots, \nu$. By the construction of $\mathcal{G}_p$ there are arithmetic progressions $P_1, \ldots, P_\nu\in\tilde{\mathcal{P}}$ of length n such that $A_i=P_i\setminus\{x_1, x_2, \ldots, x_p\}$, $i=1, \ldots, \nu$. By (5.2) we can choose a subset $\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\}$, $k\geqq n^{3/4-1/8}=n^{5/8}$ having the following property for every $1\leqq\beta<\gamma\leqq k$: either $d(P_{i_\beta})\neq d(P_{i_\gamma})$ or $P_{i_\beta}\cap P_{i_\gamma}=\emptyset$. Applying Lemma 4 to $\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\}$ we obtain a subsystem $P_{j_1}, P_{j_2}, \ldots, P_{j_t}$, $t=[n^{1/8}]$ such that $\left|\bigcup_{i=1}^{t} P_{j_i}\right|\geqq tn-n$. Observe that $\{P_{j_1}, \ldots, P_{j_t}\}$ is connected by P_0, therefore $R^*=\bigcup_{i=1}^{t} P_{j_i}\in\tilde{\mathcal{Q}}$. Since $|P_{j_i}\setminus\{x_1, x_2, \ldots, x_p\}|=|A_{j_i}|=[n^{7/8}]$, we get

$$|R^*\cap\{x_1, \ldots, x_p\}|\geqq|R^*|-\sum_{i=1}^{t}|P_{j_i}\setminus\{x_1, \ldots, x_p\}|\geqq$$

$$\geqq(tn-n)-[n^{1/8}][n^{7/8}]\geqq tn-2n.$$

Summarising,

(5.3) *there exists $R^*\in\tilde{\mathcal{Q}}$ such that $R^*\cap\{y_1, \ldots, y_p\}=\emptyset$ and $|R^*\cap\{x_1, \ldots, x_p\}|\geqq\geqq tn-2n$.*

On the other hand, we claim $|\tilde{\mathscr{Q}}|<2^{(tn-2n)-1}$. Indeed, assuming that every element of $\tilde{\mathscr{P}}$ meets at most D other, we obtain $|\tilde{\mathscr{L}}|\leqq|\tilde{\mathscr{P}}|\binom{D}{t}$. A simple argument shows $|\tilde{\mathscr{P}}|=|\tilde{\mathscr{P}}(N,n)|\leqq N^2$ and $D\leqq n^2N$, therefore by an easy computation we get

$$|\tilde{\mathscr{Q}}| \leqq N^2\binom{n^2N}{t} < 2^{(nt-2n)-1},$$

if $N\leqq 2^{n-5n^{7/8}}$ and n is large enough.

Since $\{x_{i+1}, y_i\}\subset S_i$ and $S_i\supseteqq S_{i+1}$, therefore on the red points the players fulfill the rules of the game $G(\tilde{\mathscr{Q}}, \{S_i\}_i, tn-2n)$. The second player selects his red points by the universal strategy $f_1=f_1(\tilde{\mathscr{Q}})$ and $|\tilde{\mathscr{Q}}|<2^{(tn-2n)-1}$, thus Lemma 1 yields for all $R\in\tilde{\mathscr{Q}}$ either $|R\cap\{x_1, \ldots, x_p\}|<tn-2n$ or $R\cap\{y_1, \ldots, y_p\}\neq\emptyset$, which contradicts to (5.3). Hence $v\leqq[n^{3/4}]$ is checked. This completes the proof of Theorem 2. ▌

6. Proof of Theorem 3

First we prove the upper bound in (1.5). Let $\mathbf{N}$ be an N-element set and let K_N^k denote the complete hypergraph consisting of all k-tuples of $\mathbf{N}$. For every n-element subset $H\subset\mathbf{N}$ let $K_{N|H}^k$ denote the family of all k-tuples of H and let

$$\mathscr{K}_k(N, n) = \{K_{N|H}^k\colon\ H\subset \mathbf{N}\ \textit{and}\ |H| = n\}.$$

We shall prove that the first player has a winning strategy in $G(\mathscr{K}_k(N, n))$ if $N\geqq n^{k+1}2^{\binom{n}{k}-3}$. Observe that the number of points of $\mathscr{K}_k(N, n)$ is equal to $\binom{N}{k}$, i.e. $S(\mathscr{K}_k(N, n))=\binom{N}{k}$. The $\binom{n}{k}$-uniform hypergraph $\mathscr{K}_k(N, n)$ has cardinality $\binom{N}{n}$. Since the union of two different k-tuples has cardinality at least $k+1$, thus we get

$$d_2(\mathscr{K}_k(N, n)) = \binom{N-k-1}{n-k-1}.$$

Therefore, by Theorem 1 it suffices to check that

$$\binom{N}{n} > 2^{\binom{n}{k}-3}\binom{N-k-1}{n-k-1}\binom{N}{k} \quad \text{if} \quad N\geqq n^{k+1}2^{\binom{n}{k}-3}.$$

But it is only a trivial computation, thus the upper bound in (1.5) is proved.

Finally we prove the upper bound in (1.4). Let us given an arbitrary small, but fixed $\delta>0$. Let K_N^2 denote the complete graph of N vertices. We shall explicitly prove that, if N is so large that

$$\frac{N}{\log N} \geqq \frac{4}{\delta^2}\left(\frac{1}{2}-\delta\right)^{-n+2}$$

holds, then the first player can select every edge of a complete subgraph of n vertices in K_N^2.

We need some notation. Given a graph $\mathcal{G}$, let $S(\mathcal{G})$ denote the set of its vertices. If $H\subset S(\mathcal{G})$, then $\mathcal{G}_{|H}$ denotes the subgraph induced by H. Let $d_x(\mathcal{G})$ and $d(\mathcal{G})$ denote the degree of x in $\mathcal{G}$ and the maximum degree of $\mathcal{G}$, respectively.

Let $\mathcal{K}_2^0=\mathcal{K}_2^0(N, n)$ denote the set of complete subgraphs $K_{N|H}^2$, where H is extended over all subsets of $S(K_N^2)$ having cardinality at least t_0 (t_0 will be fixed later). Using the inequality

$$2\alpha^\alpha(1-\alpha)^{1-\alpha} \geqq 1+\frac{(2\alpha-1)^2}{2}$$

we get

$$\sum_{E\in\mathcal{K}_2^0} c_\alpha^{-|E|} = \sum_{t=t_0}^{N}\binom{N}{t} c_\alpha^{-\binom{t}{2}} < 1,$$

where $\alpha=\frac{1+\delta}{2}$ and $t_0=\left[\frac{4}{\delta^2}\log N\right]$. So we can apply Lemma 3 to $\mathcal{K}_2^0$, therefore the first player has a winning strategy in the game $G(\mathcal{K}_0^2, \alpha)$.

Now consider a concrete Ramsey game on K_N^2, in which the first player is playing by the winning strategy mentioned above in $G(\mathcal{K}_0^2, \alpha)$. In the game the first player wins, therefore

$$|\mathcal{G}_{|H}| > (1-\alpha)\binom{|H|}{2} = \frac{1-\delta}{2}\binom{|H|}{2} \tag{6.1}$$

for all $H\subset S(K_N^2)$ having cardinality at least t_0, where $G\subset K_N^2$ denotes the set of edges selected by the first player during the game. Our aim is to show that the graph $\mathcal{G}$ contains a complete subgraph of n vertices. To prove it we shall apply the well-known "ramification method". From (6.1)

$$d(\mathcal{G}_{|H}) \geqq \frac{\frac{1-\delta}{2}\binom{|H|}{2}}{|H|} \geqq \left(\frac{1}{2}-\delta\right)|H|,$$

if $|H|\geqq t_0$, since $t_0\geqq\delta^{-1}$. Let N be so large that the inequality

$$\frac{N}{\log N} \geqq \frac{4}{\delta^2}\left(\frac{1}{2}-\delta\right)^{-n+2}$$

is true. Let x_1 be a vertex of $\mathcal{G}$ having maximum degree. By (6.2) $d_{x_1}(\mathcal{G})\geqq\left(\frac{1}{2}-\delta\right)N$. Let $\mathcal{G}_1$ be the subgraph induced by the neighborhood of x_1. Let x_2 be a vertex of $\mathcal{G}_1$ having maximum degree. By (6.2)

$$d_{x_2}(\mathcal{G}_1) \geqq \left(\frac{1}{2}-\delta\right) d_{x_1}(\mathcal{G}) \geqq \left(\frac{1}{2}-\delta\right)^2 N.$$

Continuing this manner we obtain vertices $x_1, \ldots, x_{n-1}$ and subgraphs $\mathcal{G}_1, \ldots, \mathcal{G}_{n-1}$ having the following properties:

a) x_i is a vertex of $\mathcal{G}_{i-1}$ having maximum degree;
b) $\mathcal{G}_i$ is the subgraph of $\mathcal{G}_{i-1}$ induced by the neighborhood of x_i;
c) $d_{x_i}(\mathcal{G}_{i-1}) \geqq \left(\frac{1}{2}-\delta\right)^i N.$

Let x_n be an arbitrary vertex of $\mathcal{G}_{n-1}$ ($\mathcal{G}_{n-1}$ is non-empty, since $d_{x_{n-1}}(\mathcal{G}_{n-2}) = |S(\mathcal{G}_{n-2})| \geqq \left(\frac{1}{2}-\delta\right)^{n-1} N > 0$), then $\{x_1, x_2, \ldots, x_n\}$ induces a complete n-graph of $\mathcal{G}$. This completes the proof of Theorem 3. ∎

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